Journal of Multivariate Analysis **78**, 235–251 (2001)
doi:10.1006/jmva.2000.1946, available online at http://www.idealibrary.com on IDEAL®

Relative Stability for Strictly Stationary Sequences

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Received September 19, 1998; published online April 6, 2001

For a nonnegative strictly stationary random sequence satisfying the "minimal" dependence condition necessary and sufficient conditions for the relative stability are found. As an application the well-known Khinchine stability result for i.i.d. random variables is proved for uniformly strong mixing sequences. © 2001 Academic Press

1. INTRODUCTION

Let $\{X_k\}_{k \in Z}$ be a strictly stationary sequence of nonnegative random variables defined on a probability space $(\Omega, \mathscr{F}, P)$. Let $S_n = \sum_{k=1}^{n} X_k$ and let $v_n \to +\infty$ be a sequence of positive numbers.

For an independent identically distributed sequence of random variables Khinchine [12] defined the relative stability of $\{X_k\}$ by the condition

$$c_n^{-1} S_n \to_P 1 \qquad \text{as} \quad n \to +\infty, \tag{1.1}$$

for some sequence $c_n \to +\infty$, and proved that Eq. (1.1) is equivalent to the slow variation of $E(X_1 \wedge x)$ [3, Theorem 8.1.1; 6, Theorem 2, VIII, Sect. 9]. In such a case normalizing sequence c_n can be defined by

$$c_n \stackrel{\text{def}}{=} \inf\{x > 0; x^{-1} E(X_1 \wedge x) < 1/n\} \tag{1.2}$$

and is 1-regularly varying [3, Theorem 1.5.12, p. 28].

W. Feller [5, Chap. X, Sect. 4] uses Eq. (1.1) as a definition of "fair" game in the case of the absence of expectations and proves that for Petersburg paradox:

$$P(X_k = 2^i) = \frac{1}{2^i}, \quad i \geqslant 1,$$

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one can take $c_n = n\log_2 n$ (for the "fair" entrance fee problem see also [17]).

Raikov [16; 7, V, Sect. 28, Theorem 4] proved that the relative stability of $\{X_k^2\}$ is equivalent to CLT. Since the Raikov Theorem remains valid for stationary martingale difference sequence (see [8, Theorem 3.2, p. 58]), the question of the relative stability (i.e., the existence of $c_n \to +\infty$, such that Eq. (1.1) holds) of an arbitrary nonnegative strictly stationary sequence is interesting. Of course, the case $EX_1 < +\infty$ is covered by the Birkhoff Theorem so we assume throughout this paper that $EX_1 = +\infty$.

The method we use to prove relative stability results was introduced by Bernstein in [1] and is as follows

For a strictly stationary random sequence $\{X_k\}$ let us define

$$\xi_j = \sum_{i=(j-1)(p+q)+1}^{jp+(j-1)q} X_i; \qquad \eta_j = \sum_{i=jp+(j-1)q+1}^{j(p+q)} X_i$$

thus we have

$$\upsilon_n^{-1} S_n = \upsilon_n^{-1} \sum_{j=1}^{k} \xi_j + \upsilon_n^{-1} \sum_{j=1}^{k} \eta_j + \upsilon_n^{-1} \sum_{i=k(p+q)+1}^{n} X_i \tag{1.3}$$

where $k(p+q) \sim n$. If the last two summands on the r.h.s. of Eq. (1.3) tend in probability to 0 then the law $\mathscr{L}(\upsilon_n^{-1} S_n)$ is asymptotically the same as the law $\mathscr{L}(\upsilon_n^{-1} \sum_1^k \xi_j)$. The latter is in turn asymptotically the same as the law $\mathscr{L}^{*k}(\upsilon_n^{-1}\xi_1)$-provided that ξ_j are *asymptotically independent*.

The above method, also known as a "big blocks by small blocks separation" requires the "dependence" condition $B(\upsilon_n)$ [10],

$$\max_{1 \leqslant k+l \leqslant n} \left| E\left(\exp\left\{it\frac{S_{k+l}}{\upsilon_n}\right\}\right) - E\left(\exp\left\{it\frac{S_k}{\upsilon_n}\right\}\right) \cdot E\left(\exp\left\{it\frac{S_l}{\upsilon_n}\right\}\right)\right| \to_n 0, \tag{1.4}$$

for some sequence $\upsilon_n \to +\infty$ of nonnegative reals and every $t \in \mathbb{R}$. Since the measure degenerated at 1 is strictly stable thus by Theorem 3.1, [10], under relative stability, 1-regular variation of normalizing sequence υ_n is equivalent to Eq. (1.4). In this sense Condition B is "minimal" and cannot be dropped.

Our criterion for relative stability will explore the notion of a sequence regularly varying in the limit.

Following [4] we shall say that the sequence of mesurable, nonnegative functions $\{f_n\}$ is $(-\gamma)$-regulary varying in the limit if there is a "rate" sequence $\{r_n\}$, $r_n \to +\infty$, such that for any sequence $\{x_n\}$, dominated by the rate sequence (i.e. such that $x_n = o(r_n)$) and $x_n \to +\infty$ we have

$$x_n^{\gamma} f_n(x_n) \to 1.$$

In the case when $\gamma = 0$ we say that $\{f_n\}$ is slowly varying in the limit.

A symmetric strictly stationary random sequence $\{X_k\}$ satisfying condition $B(\upsilon_n)$ is in the domain of attraction of symmetric strictly p-stable law, $p \in (0, 2)$, if, and only if

$$\left\{\frac{x^p}{c} P(S_n > x\upsilon_n)\right\}$$

is slowly varying in the limit for some $c > 0$ (Theorem 1, [4]). One can show ([19]) that for $p = 2$ the corresponding condition is the slow variation in the limit of

$$\left\{E\left(\frac{S_n^2}{\upsilon_n^2} \wedge x\right)\right\}.$$

The relative stability can be described in the same manner.

THEOREM 1. *Let $\{X_k\}$ be a strictly stationary sequence of nonnegative random variables for which the Condition $B(\upsilon_n)$ is fullfiled. Then the relative stability of $\{X_k\}$ is equivalent to the slow variation in the limit of the sequence of functions*

$$\left\{E\left(\frac{S_n}{\upsilon_n} \wedge x\right)\right\}.$$

While elegant Theorem 1 may be difficult to apply directly. It turns out that the version of Theorem 1 for "truncated" random variables is sometimes easier to verify. For a sequence $\{c_n\}$, $c_n \to +\infty$ let us define

$$X'_{n,k} = X_k I(X_k \leqslant c_n), \quad T'_{n,j} = \sum_{k=1}^{j} X'_{n,k}, \quad T'_n = T'_{n,n}, \quad \vartheta_n = ET'_n. \tag{1.5}$$

THEOREM 2. *Let $\{X_k\}$ be a strictly stationary sequence of nonnegative random variables. Assume that one can find a sequence $\{c_n\}$ of positive reals such that $c_n \to +\infty$ and such that ϑ_n, defined by Eq.* (1.5) *satisfies both Condition $B(\vartheta_n)$ and*

$$\vartheta_n^{-1} \sum_{k=1}^{n} X_k I(X_k > c_n) \to_P 0 \qquad \textit{as} \quad n \to +\infty. \tag{1.6}$$

Then

$$\vartheta_n^{-1} S_n \to_P 1, \qquad as \quad n \to +\infty, \tag{1.7}$$

if, and only if the sequence $\{\vartheta_n^{-1} T'_n\}_n$ *is uniformly integrable.*

Let $\mathscr{F}_k^m$ denote the σ-field generated by random variables $\{X_i; k \leqslant i \leqslant m\}$ and let us define

$$\varphi_n = \varphi_n(\{X_k\}) = \sup\{|P(B \mid A) - P(B)|; A \in \mathscr{F}_{-\infty}^0, B \in \mathscr{F}_n^\infty\}.$$

$\{X_k\}$ is φ-mixing or uniformly strong mixing if $\varphi_n \to 0$. Clearly φ-mixing sequence satisfies Condition B with arbitrary normalizing sequence $\upsilon_n \to +\infty$ (Proposition 5.2, [10]). We shall apply Theorem 1.6 to establish Khinchine stability result for φ-mixing sequences.

THEOREM 3. *Let* $\{X_k\}$ *be a strictly stationary sequence of nonnegative* φ*-mixing random variables such that* $EX_1 = \infty$. $\{X_k\}$ *is relatively stable if, and only if,* $E(X_1 \wedge x)$ *is a slowly varying function in the sense of Karamata.*

Note that in the general situation we cannot avoid some additional condition for $\{X_k\}$ (such as φ-mixing in Theorem 3).

EXAMPLE 1. Let $X_k = X$ where X is a nonnegative random variable such that $EXI(X \leqslant x)$ is a continuous slowly varying function and $EX = +\infty$. Since $EXI(X \leqslant x) \sim E(X \wedge x)$ so $c_n \sim \vartheta_n$ thus

$$\vartheta_n^{-1} S_n = \frac{X}{EXI(X \leqslant c_n)} \to_P 0,$$

and the sequence $\{X_k\}$ satisfies Condition $B(\vartheta_n)$. Observe that $\{\vartheta_n^{-1} T'_n\}$ is not uniformly integrable because for any $K > 0$

$$\vartheta_n^{-1} ET'_n I(T'_n > K\vartheta_n) = 1 - \frac{EXI(X \leqslant KEXI(X \leqslant c_n))}{EXI(X \leqslant c_n)} \to_n 1.$$

Of course, in Theorem 3 one cannot strenghten the convergence in probability to almost sure convergence.

THEOREM 4. *Let* $\{X_k\}$ *be a strictly stationary sequence of nonnegative,* φ*-mixing random variables such that* $E(X_1) = \infty$ *and* $E(X_1 \wedge x)$ *is a slowly varying function. Let* c_n *be defined by Eq.* (1.2). *Then for any* $K > 0$

$$\sum_{n=1}^{\infty} P(X_1 > Kc_n) = +\infty, \tag{1.8}$$

and

$$\limsup_n \frac{S_n}{c_n} = \limsup_n \max_{1 \leqslant k \leqslant n} \frac{X_k}{c_n} = \limsup_n \frac{X_n}{c_n} = +\infty \tag{1.9}$$

almost surely.

2. PROOFS

We begin with the following Proposition 1 parallel to Theorem 1 in [11].

PROPOSITION 1. *Let $\{X_k\}$ be a strictly stationary sequence of non-negative random variables satisfying Eq.* (1.4) *with the sequence $\upsilon_n \to +\infty$. Then*

$$\upsilon_n^{-1} S_n \to_P 1 \qquad as \quad n \to +\infty,$$

if and only if, there exists a sequence $\{r_n\}$, $r_n \to +\infty$ such that for every sequence $x_n \to +\infty$, $x_n = o(r_n)$ the following conditions are satisfied:

$$x_n P(S_n > x_n \upsilon_n) \to 0 \qquad as \quad n \to +\infty, \tag{2.10}$$

$$x_n{}^{-1} \cdot \upsilon_n^{-2} \{ES_n^2 I(S_n \leqslant x_n \upsilon_n) - (ES_n I(S_n \leqslant x_n \upsilon_n))^2\} \to 0 \qquad as \quad n \to +\infty, \tag{2.11}$$

$$\upsilon_n^{-1} ES_n I(S_n \leqslant x_n \upsilon_n) \to 1 \qquad as \quad n \to +\infty. \tag{2.12}$$

Proof of *Proposition* 1. The proof mimics the proof of Theorem 1 in [11].

Necessity. For any $n \in N$, let $\{Y_{n,i}\}_{i \in N}$ be a sequence of independent copies of $\upsilon_n^{-1} S_n$. By the hypothesis of Proposition 2.1 for each $k \in N$

$$\mathscr{L}\left(k^{-1} \sum_{i=1}^{k} Y_{n,i}\right) \to_w \delta_1 \qquad \text{as} \quad n \to +\infty.$$

Therefore there exists a sequence of real numbers $\{r_n\}$ such that $r_n \to +\infty$, and

$$\max_{1 \leqslant k \leqslant r_n} d_L\left(\mathscr{L}\left(k^{-1} \sum_{i=1}^{k} Y_{n,i}\right), \delta_1\right) \to 0 \qquad \text{as} \quad n \to +\infty,$$

where d_L denotes the Lévy metric. Let $\{k_n\}$ be a sequence of positive integers such that $\lim k_n = \infty$, $k_n = o(r_n)$, then

$$\mathscr{L}\left(k_n^{-1} \sum_{i=1}^{k_n} Y_{n,i}\right) \to_w \delta_1 \qquad \text{as} \quad n \to +\infty,$$

where $\{k_n^{-1} Y_{n,i};\, 1 \leqslant i \leqslant k_n,\, n \in N\}$ is a sequence of infinitesimal, row-wise independent random variables. Equations (2.10), (2.11), (2.12) are satisfied by the Normal Convergence Criterion for $\sigma^2 = 0$ (see [13, p. 328]).

The following lemma plays the same role as Lemma 1 in [11].

LEMMA 1. *Assume that there exists a sequence $\{r_n\}$ of positive numbers with $r_n \to +\infty$ as $n \to +\infty$, such that if $x_n \to +\infty$ and $x_n = o(r_n)$ then Eqs.* (2.10)–(2.12) *hold. If*

$$\mathscr{L}(\varsigma_n^{-1} S_n) \to_w \delta_1,$$

along subsequence $n \in Q \subseteq N$ then

$$\varsigma_n \upsilon_n^{-1} \to_n 1$$

along $n \in Q$.

Proof of Lemma 1. There exists $\{r'_n\}_{n \in Q}$, $r'_n \to \infty$ such that for $\{x'_n\}_{n \in Q}$, $x'_n \to \infty$, $x'_n = o(r'_n)$ we have

$$x'_n P(S_n > x'_n \varsigma_n) \to 0 \qquad \text{and} \qquad \varsigma_n^{-1} E S_n I(S_n \leqslant x'_n \varsigma_n) \to 1,$$

if $n \to +\infty, n \in Q$. Suppose $\lim x_n = +\infty$ and $x_n = o(r_n \wedge r'_n)$ (where r_n is taken from Eqs. (2.10)–(2.12)) and $\varsigma_n \upsilon_n^{-1} > 1 + \varepsilon$ along $n \in Q' \subseteq Q$. Integrating by parts we obtain

$$\begin{aligned}
1 &= \lim_{n \to \infty} \{x_n P(S_n > x_n \varsigma_n) + \varsigma_n^{-1} E S_n I(S_n \leqslant x_n \varsigma_n)\} \\
&= \lim_{n \to \infty} x_n \int_0^1 P(S_n > x_n \varsigma_n y)\, dy \\
&\leqslant (1+\varepsilon)^{-1} \lim_{n \to \infty} ((1+\varepsilon)\, x_n) \int_0^1 P(S_n > ((1+\varepsilon)\, x_n)\, \upsilon_n y)\, dy \\
&= (1+\varepsilon)^{-1} \lim_{n \to \infty} \{(1+\varepsilon)\, x_n P(S_n > (1+\varepsilon)\, x_n \upsilon_n) \\
&\qquad + \upsilon_n^{-1} E S_n I(S_n \leqslant (1+\varepsilon)\, x_n \upsilon_n)\} \\
&= (1+\varepsilon)^{-1}
\end{aligned}$$

because the sequence $(1+\varepsilon)\, x_n$ is dominated by the rate sequence $r_n \wedge r'_n$. Hence we have

$$\limsup_{n \in Q'} \frac{\varsigma_n}{\upsilon_n} \leqslant 1.$$

Since ς and υ can be exchanged in the above, Lemma 2.4 holds. ∎

Sufficiency. Let $\{Y_{n,j},\ j \in N, n \in N\}$ be a sequence of row-wise i.i.d. random variables such that $\mathscr{L}(Y_{n,j}) = \mathscr{L}(\upsilon_n^{-1} S_n)$. By the Normal Convergence Criterion for $\sigma^2 = 0$ (see [13], p. 328) Eqs. (2.10), (2.11) and (2.12) imply

$$\mathscr{L}\left(k_n^{-1} \sum_{j=1}^{k_n} Y_{n,j}\right) \to_w \delta_1, \qquad \text{as} \quad n \to +\infty \tag{2.13}$$

for any sequence $\{k_n\}_{n \in N}$ of positive integers such that $\lim k_n = +\infty$ and $k_n = o(r_n)$.

The estimate, valid for fixed $k \in N$;

$$\begin{aligned}
&\left| E\left(\exp\left\{it\upsilon_{kn}^{-1} \sum_{j=1}^{kn} X_j\right\}\right) - \left(E\left(\exp\left\{it\upsilon_{kn}^{-1} S_n\right\}\right)\right)^k \right| \\
&= \left| E\left(\exp\left\{it\upsilon_{nk}^{-1} \sum_{j=1}^{k} \left(\sum_{i=1}^{n} X_{n(j-1)+i}\right)\right\}\right) - \prod_{j=1}^{k} E\left(\exp\left\{it\upsilon_{kn}^{-1} S_n\right\}\right) \right| \\
&\leqslant \sum_{l=1}^{k-1} \left| E\left(\exp\left\{it\upsilon_{nk}^{-1} \sum_{j=1}^{k-l+1} \left(\sum_{i=1}^{n} X_{n(j-1)+i}\right)\right\}\right) \cdot \prod_{j=1}^{l-1} E\left(\exp\left\{it\upsilon_{nk}^{-1} S_n\right\}\right) \right. \\
&\left. - E\left(\exp\left\{it\upsilon_{kn}^{-1} \sum_{j=1}^{k-l} \left(\sum_{i=1}^{n} X_{n(j-1)+i}\right)\right\}\right) \cdot \prod_{j=1}^{l} E\left(\exp\left\{it\upsilon_{kn}^{-1} S_n\right\}\right) \right| \\
&\leqslant \sum_{l=1}^{k-1} \left| E\left(\exp\left\{it\frac{S_{(k-l+1)n}}{\upsilon_{nk}}\right\}\right) - E\left(\exp\left\{it\frac{S_{(k-l)n}}{\upsilon_{kn}}\right\}\right) \cdot E\left(\exp\left\{it\frac{S_n}{\upsilon_{kn}}\right\}\right) \right| \\
&\leqslant k \max_{1 \leqslant p+q \leqslant kn} \left| E\left(\exp\left\{it\frac{S_{p+q}}{\upsilon_{kn}}\right\}\right) - E\left(\exp\left\{it\frac{S_p}{\upsilon_{kn}}\right\}\right) \cdot E\left(\exp\left\{it\frac{S_q}{\upsilon_{kn}}\right\}\right) \right|
\end{aligned}$$

and Eq. (1.4) yield

$$\mathscr{L}\left(\frac{S_{k_n n}}{\upsilon_{k_n n}}\right) \sim \mathscr{L}^{*k_n}\left(\frac{S_n}{\upsilon_{k_n n}}\right), \tag{2.14}$$

for $k_n \to +\infty$ and new “rate” sequence $\{r'_n\}$.

We shall prove that Eq. (2.10) gives tightness of $\{v_{k_n n}^{-1} S_{k_n n}\}$ and Eq. (2.12) implies that any weak limit is not degenerated at 0. To see this assume first that $\{v_{k_n n}^{-1} S_{k_n n}\}$ is not tight. Then we have

$$\lim_{k \to \infty} \limsup_{n \to \infty} P(v_n^{-1} S_n > k) \geqslant \lim_{k \to \infty} \limsup_{n \to \infty} P(v_{k_n n}^{-1} S_{k_n n} > k) \geqslant \eta > 0.$$

Let $N_l > N_{l-1}$ be a sequence of positive integers such that

$$P(v_{N_l}^{-1} S_{N_l} > l) > \tfrac{1}{2}\eta$$

and the index l is such that $r_n > l^2$ for $n \geqslant N_l$. Define $k_n = l$ for $N_l \leqslant n < N_{l+1}$. Observe that $\{k_n\}$ satisfies

$$\frac{k_n}{r_n} \leqslant \frac{1}{k_n} \to 0$$

and

$$\liminf_{l \to \infty} k_{N_l} P(v_{N_l}^{-1} S_{N_l} > k_{N_l}) \geqslant \liminf_{l \to \infty} k_{N_l} \cdot \tfrac{1}{2}\eta = +\infty$$

which contradicts Eq. (2.10) and thus proves the tightness of $\{v_n^{-1} S_n\}$ (and $\{v_{k_n n}^{-1} S_{k_n n}\}$). To verify that each weak limit of $\{v_{k_n n}^{-1} S_{k_n n}\}$ is not degenerated at 0 it is enough to verify the same for the sequence $\{v_n^{-1} S_n\}$. Assume that this is not the case, i.e. that along some subsequence n' we have

$$\mathscr{L}(v_{n'}^{-1} S_{n'}) \to_w \delta_0.$$

Then there exists a sequence $k_{n'} = o(r_{n'})$ such that

$$\lim_{n' \to \infty} v_{n'}^{-1} E(S_{n'} I(S_{n'} \leqslant k_{n'} v_{n'})) = 0$$

which contradicts Eq. (2.12) and proves that any weak limit of $\{v_n^{-1} S_n\}$ is not degenerated at 0.

Assume that Z is a limit along the subsequence $\{k_{n'}\}$ such that $\mathscr{L}(Z) \neq \delta_0$ and

$$\mathscr{L}\left(\frac{S_{k_{n'} n'}}{v_{k_{n'} n'}}\right) \to_w \mathscr{L}(Z). \tag{2.15}$$

By Eq. (2.14) we have

$$\mathscr{L}\left(\frac{v_{n'}}{v_{k_{n'} n'}} \sum_{j=1}^{k_{n'}} Y_{n',j}\right) \to_w \mathscr{L}(Z). \tag{2.16}$$

Suppose that along subsubsequence $\{n''\} \subset \{n'\}$ we have

$$\frac{k_{n''} v_{n''}}{v_{k_{n''} n''}} \to_{n''} c.$$

Now if $c = 0$ then by Eqs. (2.16), (2.13) we have that $\mathscr{L}(Z) = \delta_0$. If $c = +\infty$ then Eqs. (2.13), (2.16), (2.14) imply that the sequence $\{v_{k_n n}^{-1} S_{k_n n}\}$ is not tight—which is not possible as we have proved above. So we may assume that $0 < c < +\infty$ and

$$0 < c_1 = \liminf_{n'} \frac{k_{n'} v_{n'}}{v_{k_{n'} n'}} < \limsup_{n'} \frac{k_{n'} v_{n'}}{v_{k_{n'} n'}} = c_2 < +\infty.$$

By this and Eqs. (2.16), (2.13) we have for $0 < c_1 < c_2 < +\infty$

$$\mathscr{L}(c_1^{-1} Z) = \mathscr{L}(c_2^{-1} Z),$$

hence for $c = \frac{c_1}{c_2} < 1$

$$\mathscr{L}(Z) = \mathscr{L}(cZ) = \mathscr{L}(c^2 Z) = \cdots = \cdots = \lim_{n \to \infty} \mathscr{L}(c^n Z) = \delta_0.$$

The above again is not possible since we have proved that $\mathscr{L}(Z) \neq \delta_0$ so finally $\mathscr{L}(Z) = \delta_c$ for some $0 < c < +\infty$. By Eq. (2.15) let $r''_n \to +\infty$ be such that

$$\mathscr{L}\left(l_{n'}^{-1} \sum_{j=1}^{l_{n'}} Y_{n' k_{n'}, j} \right) \to_w \mathscr{L}(Z),$$

for every $l_{n'} \to +\infty$, $l_{n'} = o(r''_{n'})$. In particular, if $l_{n'} = o(r''_{n'} \wedge r_{n' k_{n'}})$, Eqs. (2.10)–(2.12) imply $\mathscr{L}(Z) = \delta_1$. This proves that $c = 1$ and so

$$v_{n \cdot k_n}^{-1} v_n k_n \to 1 \qquad \text{as} \quad n \to +\infty.$$

This, Eqs. (2.13) and (2.14) imply

$$\mathscr{L}((v_{nk_n})^{-1} S_{nk_n}) \to_w \delta_1 \qquad \text{as} \quad n \to +\infty, \tag{2.17}$$

for any $k_n \to \infty$, $k_n = o(r_n \wedge r'_n)$.

In order to finish the proof we need to show that one can approximate S_n by $S_{[(n + k_n) k_n^{-1}] k_n}$. Let us define $a_n = \min_{i \geqslant n} v_i$ and choose the sequence $\{m_n\}$ of positive integers such that, $\lim m_n = \infty$ and

$$\max_{1 \leqslant j \leqslant m_n} \frac{S_j}{a_n} \to_P 0 \qquad \text{as} \quad n \to +\infty.$$

For the rate sequence $r_n \wedge r'_n \wedge m_n$ we have

$$P(v^{-1}_{[(n+k_n)\,k_n^{-1}]\,k_n}(S_{[(n+k_n)\,k_n^{-1}]\,k_n} - S_n) > \varepsilon)$$
$$\leqslant P(\max_{1 \leqslant j \leqslant k_n} a_n^{-1} S_j > \varepsilon) \to 0$$

as $n \to +\infty$. By Lemma 1 we have to prove that

$$\mathscr{L}\left(v^{-1}_{[(n+k_n)\,k_n^{-1}]\,k_n} S_{[(n+k_n)\,k_n^{-1}]\,k_n}\right) \to_w \delta_1 \qquad \text{as} \quad n \to +\infty.$$

In order to use Eq. (2.17) we have to find k_n and s_n such that

$$\frac{k_n}{s_{[(n+k_n)\,k_n^{-1}]}} \to 0 \qquad \text{as} \quad n \to +\infty.$$

Define the rate $s_n = r_n \wedge r'_n \wedge m_n \wedge \sqrt{n}$ and observe that $s_n \nearrow +\infty$ (r_n, r'_n, m_n can be choosen increasing), $ns_n^{-1} \to +\infty$ and if $k_n = o(s_{[ns_n^{-1}]})$ and $k_n \to +\infty$ then,

$$\frac{k_n}{s_{[(n+k_n)\,k_n^{-1}]}} \leqslant \frac{k_n}{s_{[nk_n^{-1}]}} \leqslant \frac{k_n}{s_{[ns_n^{-1}]}} \to 0 \qquad \text{as} \quad n \to +\infty.$$

This completes the proof of Proposition 1. ∎

Proof of Theorem 1.

Necessity. Since by Lebesgue dominated convergence theorem $E(\frac{S_n}{v_n} \wedge m) \to 1$ for every $m > 1$ one can apply Corollary 8.5, [10].

Sufficiency. Assume that for some sequence $\{x_n\}$, $x_n \to +\infty$

$$\int_0^{x_n} P(S_n > yv_n)\, dy = E\left(\frac{S_n}{v_n} \wedge x_n\right) \to 1.$$

Let $y_n = o(x_n)$, $y_n \to \infty$, then

$$\int_{y_n}^{x_n} P(S_n > yv_n)\, dy = \left\{E\left(\frac{S_n}{v_n} \wedge x_n\right) - E\left(\frac{S_n}{v_n} \wedge y_n\right)\right\} \to_n 0.$$

By this we find that

$$x_n\left(1 - \frac{y_n}{x_n}\right) P\,(S_n > x_n v_n) \leqslant \int_{y_n}^{x_n} P(S_n > yv_n)\, dy \to 0,$$

hence

$$x_n P(S_n > x_n v_n) \to 0, \tag{2.18}$$

so Eq. (2.10) holds. On the other hand by the following identity

$$E\frac{S_n}{v_n} I\left(\frac{S_n}{v_n} \leqslant x_n\right) = -x_n P(S_n > x_n v_n) + \int_0^{x_n} P(S_n > y v_n)\, dy$$

for $x_n = o(r_n)$, $x_n \to \infty$ we get

$$E\left(\frac{S_n}{v_n} I\left(\frac{S_n}{v_n} \leqslant x_n\right)\right) \to 1,$$

so Eq. (2.12) holds. Thus it remains to establish Eq. (2.11). Since

$$E\left(\frac{S_n}{v_n} I\left(\frac{S_n}{v_n} \leqslant x_n\right)\right) \to 1,$$

thus

$$x_n^{-1} \left(E\left(\frac{S_n}{v_n} I\left(\frac{S_n}{v_n} \leqslant x_n\right)\right)\right)^2 \to 0,$$

so we need to show that

$$x_n^{-1} E\left(\frac{S_n^2}{v_n^2} I\left(\frac{S_n}{v_n} \leqslant x_n\right)\right) \to 0.$$

For this observe that

$$E\left(\frac{S_n^2}{v_n^2} I\left(\frac{S_n}{v_n} \leqslant x_n\right)\right) = -x_n^2\, P(S_n > x_n v_n) + 2\int_0^{x_n} yP(S_n > y v_n)\, dy,$$

so by Eq. (2.18) it suffices to show that

$$x_n^{-1} \int_0^{x_n} yP(S_n > y v_n)\, dy \to 0. \tag{2.19}$$

We have

$$\int_0^{x_n} yP(S_n > y v_n)\, dy$$
$$= \int_0^{x_n} y\, dE\left(\frac{S_n}{v_n} \wedge y\right) = x_n\, E\left(\frac{S_n}{v_n} \wedge x_n\right) - \int_0^{x_n} E\left(\frac{S_n}{v_n} \wedge y\right) dy,$$

by integration by parts. Now by Theorem 8.2, [10] and assumptions of the theorem we obtain

$$x_n^{-1} \int_0^{x_n} E\left(\frac{S_n}{v_n} \wedge y\right) dy \to 1,$$

so Eq. (2.19) holds. Thus from Eq. (2.19) follows Eq. (2.11) and Theorem 1 is proved. ∎

Proof of Theorem 2.

Necessity. Eqs. (1.6) and (1.7) imply $\vartheta_n^{-1} T'_n \to_P 1$. Uniform integrability of $\{\vartheta_n^{-1} T'_n\}$ follows from Theorem 5.4, [2], p. 32.

Sufficiency. Let $\{\vartheta_n^{-1} T'_n\}$ be uniformly integrable, so by Lemma 1, [11], we obtain for $x_n \to \infty$, $x_n = o(r_n)$

$$E \frac{T'_n}{\vartheta_n} I\left(\frac{T'_n}{\vartheta_n} > x_n\right) = x_n P(T'_n > x_n \vartheta_n) + \int_{x_n}^{+\infty} P(T'_n > y\vartheta_n)\, dy \to 0,$$

which proves that

$$x_n P(T'_n > x_n \vartheta_n) \to_n 0.$$

By this and the identity

$$E \frac{T'_n}{\vartheta_n} I\left(\frac{T'_n}{\vartheta_n} \leqslant x_n\right) = -x_n P(T'_n > x_n \vartheta_n) + \int_0^{x_n} P(T'_n > y\vartheta_n)\, dy \to 1,$$

we have

$$\int_0^{x_n} P(T'_n > y\vartheta_n)\, dy \to 1.$$

Since for every $k \in N$ and $\varepsilon > 0$ by Eq.(1.6)

$$\int_0^k P(S_n - T'_n > \varepsilon y \vartheta_n)\, dy \to 0,$$

thus there exists "rate" r'_n, $r'_n \leqslant r_n$ such that for $x_n = o(r'_n)$

$$\begin{aligned} 0 &\leqslant \int_0^{x_n} P(S_n > y\vartheta_n)\, dy - \int_0^{x_n} P(T'_n > y\vartheta_n)\, dy \\ &\leqslant \int_0^{x_n} P(S_n - T'_n > \varepsilon y \vartheta_n)\, dy + \frac{1}{1-\varepsilon} \int_0^{(1-\varepsilon)\, x_n} P(T'_n > y\vartheta_n)\, dy \\ &\quad - \int_0^{x_n} P(T'_n > y\vartheta_n)\, dy \to \frac{1}{1-\varepsilon} - 1. \end{aligned}$$

Hence we proved that

$$\left\{E\left(\frac{S_n}{\vartheta_n}\wedge x\right)\right\}=\left\{\int_0^x P(S_n>y\vartheta_n)\,dy\right\}$$

is slowly varying in the limit and by Theorem 1 Eq. (1.7) holds. ∎

Proof of Theorem 3.

Neccesity. Assume that $\upsilon_n^{-1}S_n\rightarrow_P 1$. We shall prove first that

$$nP(X_1>2\upsilon_n)\rightarrow_n 0. \tag{2.20}$$

Let p be such that $\varphi_p<1$. By Proposition 3.1, [15] for $n>p$ we have

$$P(\max_{1\leqslant k\leqslant[n/p]} X_{kp}>x)\geqslant(1-\varphi_p)\,P(\max_{1\leqslant k\leqslant[n/p]} X_{kp}^*>x),$$

where $\{X_k^*\}$ is an i.i.d. sequence such that $\mathscr{L}(X_1)=\mathscr{L}(X_k^*)$. Hence for $n>p$

$$P(\max_{1\leqslant k\leqslant n} X_k\leqslant x)\leqslant 1-(1-\varphi_p)\,P(\max_{1\leqslant k\leqslant[n/p]} X_{kp}^*>x).$$

This and nonnegativity of X_k and Eq. (1.1) imply

$$\begin{aligned}1=\liminf_n P(S_n\leqslant 2\upsilon_n)&\leqslant\liminf_n P(\max_{1\leqslant k\leqslant n} X_k\leqslant 2\upsilon_n)\\&\leqslant\liminf_n\{1-(1-\varphi_p)\,P(\max_{1\leqslant k\leqslant[n/p]} X_k^*>2\upsilon_n)\}\\&\leqslant\varphi_p+(1-\varphi_p)\liminf_n P^{[n/p]}(X_1\leqslant 2\upsilon_n).\end{aligned}$$

Thus by the elementary inequality

$$x^n\leqslant\exp\{-n(1-x)\},\qquad 0<x\leqslant 1,$$

we have that

$$1-\varphi_p\leqslant(1-\varphi_p)\exp\left\{-\limsup_n\left[\frac{n}{p}\right]P(X_1>2\upsilon_n)\right\},$$

hence Eq. (2.20) holds. $\upsilon_n^{-1}S_n\rightarrow_P 1$ and $\upsilon_n\rightarrow_n+\infty$ imply

$$\frac{\upsilon_{n+1}}{\upsilon_n}\rightarrow_n 1. \tag{2.21}$$

By Eqs. (2.20) and (1.1) with $c_n = \upsilon_n$ we have for $0<\varepsilon<1$

$$1=\liminf_n P\left(\sum_{k=1}^n X_k I(X_k \leqslant 2\upsilon_n) > (1-\varepsilon)\,\upsilon_n\right) \leqslant \liminf_n \frac{nEX_1 I(X_1 \leqslant 2\upsilon_n)}{(1-\varepsilon)\,\upsilon_n}.$$

Hence

$$\frac{2\upsilon_n P(X_1>2\upsilon_n)}{EX_1 I(X_1 \leqslant 2\upsilon_n)} = \frac{2nP(X_1>2\upsilon_n)}{\dfrac{n}{\upsilon_n} EX_1 I(X_1 \leqslant 2\upsilon_n)} \to_n 0.$$

Thus Eq. (2.21) implies that for $2\upsilon_n \leqslant x < 2\upsilon_{n+1}$

$$\frac{xP(X_1>x)}{EX_1 I(X_1 \leqslant x)} \leqslant \frac{2\upsilon_{n+1} P(X_1>2\upsilon_n)}{EX_1 I(X_1 \leqslant 2\upsilon_n)} \to_n 0,$$

which proves the slow variation of $EX_1 I(X_1 \leqslant x)$ and $E(X_1 \wedge x)$ [6, Theorem 2, VIII, Sect. 9]. Observe that in the proof we used the condition $\lim_n \varphi_n < 1$ only.

Sufficiency. Let c_n be defined as in Eq. (1.2). We have for any $\varepsilon>0$

$$P\left(\vartheta_n^{-1} \sum_{k=1}^n X_k I(X_k > c_n) > \varepsilon\right) \leqslant nP(X_1>c_n)$$
$$= \frac{c_n P(X_1>c_n)}{c_n P(X_1>c_n) + EX_1 I(X_1 \leqslant c_n)} \cdot \frac{n}{c_n} E(X_1 \wedge c_n) \to_n 0,$$

by Theorem 2, VIII, Sect. 9 of [6] and the definition of c_n so Eq. (1.6) holds. According to Theorem 2 it is enough to verify uniform integrability of $\{\vartheta_n^{-1} T'_n\}$. Observe that

$$\frac{xP(X_1>x)}{E(X_1 \wedge x)} \leqslant \frac{2(E(X_1 \wedge x) - E(X_1 \wedge (2^{-1}x)))}{E(X_1 \wedge x)} \to 0,$$

as $x \to +\infty$ so by

$$E(X_1 \wedge x) = xP(X_1>x) + EX_1 I(X_1 \leqslant x),$$

we get $EX_1 I(X_1 \leqslant x) \sim E(X_1 \wedge x)$ and $\vartheta_n \sim c_n$. Thus the sequence

$$\{\max_{1 \leqslant i \leqslant n} \vartheta_n^{-1} X_i I(X_i \leqslant c_n)\}_n$$

is bounded a.s. and so is uniformly integrable. By the Markov inequality and stationarity we have

$$\max_{1\leqslant i\leqslant n} P((T'_n - T'_{n,i}) > \vartheta_n 2^{-1} bx) \leqslant \max_{1\leqslant i\leqslant n} (\vartheta_n bx)^{-1}\, 4E(T'_{n,i}) = \frac{4}{bx}.$$

By φ-mixing and the fact that r.h.s. of the above can be arbitrary small for large x and b we obtain that there exist $a_0 > 0$, $b > 0$, $p \in N$ and $\eta < \frac{1}{2}$, such that the condition

$$\varphi_p + \max_{1\leqslant i\leqslant n} P(|T'_n - T'_{n,i}| \geqslant \vartheta_n 2^{-1} b a_0) \leqslant \eta < 1.$$

is fullfiled together with

$$\frac{\eta(1+2b)}{(1-\eta)} < 1. \tag{2.22}$$

By Lemma 3.1, [14] we get for $n > p$ i $x > a_0$

$$P(T'_n > (1+2b)\, x\vartheta_n) \leqslant (1-\eta)^{-1}\, P(\max_{1\leqslant i\leqslant n} X'_{n,i} > (2p)^{-1}\, bx\vartheta_n)$$
$$+ \eta\cdot(1-\eta)^{-1}\, P(T'_n > x\vartheta_n).$$

Integrating both sides and using well known formula

$$EXI(X > x) = xP(X > x) + \int_x^{+\infty} P(X > y)\, dy,$$

we conclude that for $K > a_0$

$$E\frac{T'_n}{\vartheta_n} I\left(\frac{T'_n}{\vartheta_n} > (1+2b)\, K\right) \leqslant \frac{\eta(1+2b)}{(1-\eta)}\, E\frac{T'_n}{\vartheta_n} I\left(\frac{T'_n}{\vartheta_n} > K\right)$$
$$+ \frac{2p(1+2b)}{b(1-\eta)}\, E\max_{1\leqslant i\leqslant n} \frac{X'_{n,i}}{\vartheta_n} I\left(\max_{1\leqslant i\leqslant n} \frac{X'_{n,i}}{\vartheta_n} > (2p)^{-1}\, bK\right).$$

Taking $\sup_n$ over both sides and using the fact that

$$\sup_n E\frac{T'_n}{\vartheta_n} I\left(\frac{T'_n}{\vartheta_n} > K\right)$$

is decreasing function of K and going with $K \to +\infty$ we get from Eq. (2.22) that $\vartheta_n^{-1} T'_n$ is uniformly integrable. ∎

Proof of Theorem 4. We shall establish firstly that, if $EX_1 = +\infty$ and $E(X_1 \wedge x)$ is 1-regularly varying function then, the condition (1.8) is satisfied. Assume that this is not true, i.e., for some $K > 0$

$$\sum_{n=1}^{\infty} P(X_1 > Kc_n) < +\infty.$$

Then by the fact that $\{c_n\}$ is 1-regularly varying and by Lemma 3.2.4 ([18], p. 131), we obtain

$$\sum_{n=1}^{\infty} P(X_1 > Kc_n) < +\infty,$$

for every $K > 0$. By Feller's Theorem (see Theorem 3.2.5, [18], p. 132) for independent random variables $\{X_k^*\}$ such that, $\mathscr{L}(X_k^*) = \mathscr{L}(X_1)$ we have

$$\limsup_n \frac{\sum_{k=1}^n X_k^*}{c_n} = 0,$$

a.s. This contradicts Eq. (1.1) which holds by Khinchine's Theorem. Thus for every $K > 0$

$$\sum_{n=1}^{\infty} P(X_1 > Kc_n) = +\infty.$$

By Eq. (1.8) the converse Borel–Cantelli lemma for φ-mixing sequences (see Lemma 1.1.2′, [9], p. 3) can be applied. Thus we have for φ_p and $K > 0$

$$P\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \{X_n > Kc_n\}\right) \geqslant P\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \{X_{np} > Kc_{np}\}\right) \geqslant 1 - \varphi_p,$$

which by the choice of K and for $\varphi_p \to_p 0$ proves the last equality in Eq. (1.9) while the nonnegativity of X_k yields the rest. ∎

ACKNOWLEDGMENTS

The author thanks Professor Adam Jakubowski for useful comments and the referee for carefully reading the manuscript and for his remarks which improved the presentation of this paper.

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